Govt. Women Engineering college, Ajmer

MODEL ANSWER PAPER

B.Tech IV semester

Branch :- ECE

Sub :- Optimization Techniques

Q1. Solve the LPP by simplex method

$$\text{Min } Z = 5x_1 + 2x_2$$

s.t.

$$3x_1 + x_2 = 4$$

$$2x_1 + x_2 \geq 3$$

$$x_1 + 2x_2 \leq 3$$

and $x_1, x_2 \geq 0$

Solution :- First write the LPP into standard form as

$$\text{Max } (-Z) = -5x_1 - 2x_2 + 0x_3 + 0x_4$$

s.t.

$$3x_1 + x_2 = 4$$

$$2x_1 + x_2 - x_3 = 3$$

$$x_1 + 2x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

where x_3 is surplus variable and x_4 is slack variable

Now coefficient Matrix is $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Here we are not getting Basis, so we introduce artificial variables x_5 and x_6 and assign '-M' cost to these variables.

$$\text{Max } (-Z) = -5x_1 - 2x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$$

$$3x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 4$$

$$2x_1 + x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 3$$

$$x_1 + 2x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 = 3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Now Basic variables are x_4, x_5 and x_6.

First simplex table

			c_j	-5	-2	0	0	-M	-M	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
-M	α_5	x_5	4	$\boxed{3}$	1	0	0	1	0	→
-M	α_6	x_6	3	2	1	-1	0	0	1	
0	α_4	x_4	3	1	2	0	1	0	0	
	$z_j - c_j$			-5M +5	-2M +2	M	0	0	0	
				↑						

Since $z_1 - c_1$ is most negative $\therefore$ x_1 will be enentering vector

key element $= \text{Min}\left[\frac{4}{3}, \frac{3}{2}, \frac{3}{1}\right] = \frac{4}{3}$

$\therefore$ $y_{11} = 3$ is key element and x_5 will be departing vector. since x_5 is anartificial variable so we drop it coloumn in next table

			c_j	-5	-2	0	0	-M	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_6	
-5	α_1	x_1	4/3	1	1/3	0	0	0	
-M	α_6	x_6	1/3	0	$\boxed{1/3}$	-1	0	1	→
0	α_4	x_4	5/3	0	5/3	0	1	0	
	$z_j - c_j$			0	$\frac{-M}{3}+\frac{1}{3}$	M	0	0	

as $z_2 - c_2$ is most negative $\therefore$ x_2 will be entering vector

key Element $= \text{Min}\left[\frac{4}{3}/\frac{1}{3}, \frac{1}{3}/\frac{1}{3}, \frac{5}{3}/\frac{5}{3}\right] = \text{Min}[4, 1, 1]$

as minimum ratio is same for y_{22} and y_{23} so any one can be taken as key element we will take $y_{22} = \frac{1}{3}$ as key element, so x_6 will be departing vector and it is an artifical variable so we drop its coloumn in next table

		c_j		-5	-2	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
-5	α_1	x_1	1	1	0	1	0
-2	α_2	x_2	1	0	1	-3	0
0	α_4	x_4	0	0	0	5	1
	$z_j - c_j$			0	0	1	0

$\therefore$ all $z_j - c_j \geq 0$ so the solution optimal

$x_1 = 1, \quad x_2 = 1, \quad x_4 = 0$

$\text{Max}(z) = -5 \times 1 - 2 \times 1 = -7$

$\text{Min } z = \text{Max}(-z) = 7$

Q2. Solve the minimum assigment Problem

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

Solution :- Substract least element of each row from every element of the corresponding row

	I	II	III	IV	V
A	3	9	0	8	12
B	3	1	6	0	9
C	1	4	3	0	4
D	4	7	0	11	9
E	4	0	2	1	5

Substract least element of each coloumn from every element of the corresponding coloumn

	I	II	III	IV	V
A	2	9	0	8	8
B	2	1	6	0	5
C	0	4	3	0	0
D	3	7	0	11	5
E	3	0	2	1	1

apply assignment Procedure

	I	II	III	IV	V
A	2	9	$\boxed{0}$	8	8
B	2	1	6	$\boxed{0}$	5
C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$
D	3	7	$\cancel{0}$	11	5
E	3	$\boxed{0}$	2	1	1

As fourth row does not have any assignment therefore it is not optimum level. Now we draw straight lines

	I	II	III	IV	V
A	2	9	$\boxed{0}$	8	8
B	2	1	6	$\boxed{0}$	5
C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$
D	3	7	$\cancel{0}$	11	5
E	3	$\boxed{0}$	2	1	1

least uncovered element is 2, so we subtract 2 from uncovered elements and add to the elements which lies at the intersection of lines and again apply assignment procedure

	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	6	6
B	2	1	8	$\boxed{0}$	5
C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
D	1	5	$\boxed{0}$	9	3
E	3	$\boxed{0}$	4	1	1

as each row and each coloumn has assignment so it is optimum ~~test~~ level. Solution is

A → I, B → IV, C → V, D - III, E - II

Minimum cost is 11 + 6 + 16 + 17 + 10 = 60

Q3 Use the revised simplex method to solve the LPP

$$\text{Max. } Z = 5x_1 + 3x_2$$
$$4x_1 + 5x_2 \geq 10$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

Solution :- first write the LPP into standard form

$$\text{Max. } Z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$$
$$5x_1 + 2x_2 + x_3 = 10$$
$$3x_1 + 8x_2 + x_4 = 12$$
$$4x_1 + 5x_2 - x_5 = 10$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

where x_3, x_4 are slack variables and x_5 is surplus variables

$$A = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 \\ 4 & 5 & 0 & 0 & -1 \end{bmatrix}$$

as we are not getting basis so we add artificel variable in third constraint

$$\text{Max. } Z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$$
$$5x_1 + 2x_2 + x_3 = 10$$
$$3x_1 + 8x_2 + x_4 = 12$$
$$4x_1 + 5x_2 - x_5 + x_6 = 10$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Now

$$\hat{A} = \begin{bmatrix} 5 & 2 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & -1 & 1 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 10 \\ 12 \\ 10 \\ 0 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & M & 1 \end{bmatrix} \qquad \hat{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -M & 1 \end{bmatrix}$$

Now

Net evaluation $= z_j - c_j = [C_B B^{-1} \;\; 1] \; \hat{A}$

$$= [0 \;\; 0 \;\; -M \;\; 1] \begin{bmatrix} 5 & 2 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & -1 & 1 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= [-4M-5, \; -5M-3, \; 0, \; 0, \; M, \; 0]$$

as most negative $z_j - c_j$ is $z_2 - c_2$ therefore x_2 will be entering vector

Now $\hat{x}_2 = \hat{B}^{-1} \hat{a}_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -M & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 5 \\ -5M-3 \end{bmatrix}$$

$$\hat{x}_B = \hat{B}^{-1} b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -M & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 10 \\ -10M \end{bmatrix}$$

First Revised Simplex table is

	$\hat{B}^{-1}$				$\hat{x}_B$	$\hat{x}_2$	Mini ratio
x_3	1	0	0	0	10	2	5
x_4	0	1	0	0	12	$\boxed{8}$	3/2 →
x_6	0	0	1	0	10	5	2
	0	0	-M	1	-10M	-5M-3	

$\hat{x}_{22}$ is key element and x_4 is departing vector

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 1 & -1/4 & 0 & 0 \\ 0 & 1/8 & 0 & 0 \\ 0 & -5/8 & 1 & 0 \\ 0 & \frac{5M+3}{8} & -M & 1 \end{bmatrix}$$

Net evaluation $z_j - c_j = [c_B B^{-1} \; 1] \hat{A}$

$$= \begin{bmatrix} 0 & \frac{5M+3}{8} & -M & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & -1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \cancel{\left[\frac{15M+9}{8}, -4M-5, \frac{40M+24}{8}, -8M-13\right.}$$

$$= \left[\frac{-17M-31}{8}, \; 0, \; 0 \; \frac{5M+3}{8}, \; M, \; 0\right]$$
$\uparrow$

∴ all $z_j - c_j \not\geq 0$ and most negative is $z_1 - c_1$, ∴ x_1 will be entering vector

$$\hat{x}_1 = \hat{B}^{-1} a_1 = \begin{bmatrix} 1 & -1/4 & 0 & 0 \\ 0 & 1/8 & 0 & 0 \\ 0 & -5/8 & 1 & 0 \\ 0 & \frac{5M+3}{8} & -M & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 17/4 \\ 3/8 \\ 17/8 \\ \frac{-17M-31}{8} \end{bmatrix}$$

$$\hat{x}_B = \hat{B}^{-1} b = \begin{bmatrix} 1 & -1/4 & 0 & 0 \\ 0 & 1/8 & 0 & 0 \\ 0 & -5/8 & 1 & 0 \\ 0 & \frac{5M+3}{8} & -M & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3/2 \\ 5/2 \\ \frac{-5M+9}{2} \end{bmatrix}$$

II^{nd} revised simplex table is:

	$\hat{B}^{-1}$				$\hat{x}_B$	$\hat{x}_1$	mini ratio
x_3	1	$-1/4$	0	0	7	$17/4$	$28/17$
x_2	0	$1/8$	0	0	$3/2$	$3/8$	4
x_6	0	$-5/8$	1	0	$5/2$	$\boxed{17/8}$	$20/17$ →
	0	$\frac{5M+3}{8}$	$-M$	1	$\frac{-5M+9}{2}$	$\frac{-17M-31}{8}$	

y_{31} is key element and x_6 is departing vector

Now

$$\hat{B}^{-1} = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 4/17 & -3/17 & 0 \\ 0 & -5/17 & 8/17 & 0 \\ 0 & -13/17 & 31/17 & 1 \end{bmatrix}$$

Net evaluation $z_j - c_j = \begin{bmatrix} 0 & \frac{-13}{17} & \frac{31}{17} & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & -1 & 1 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{-13}{17} & \frac{-31}{17} & \frac{31}{17} + M \end{bmatrix}$$

x_5 is entering vector

$$\hat{x}_5 = \hat{B}^{-1} a_5 = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 4/17 & -3/17 & 0 \\ 0 & -5/17 & 8/17 & 0 \\ 0 & -13/17 & 31/17 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/17 \\ -8/17 \\ -31/17 \end{bmatrix}$$

$$\hat{x}_B = \hat{B}^{-1} \hat{b} = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 4/17 & -3/17 & 0 \\ 0 & -5/17 & 8/17 & 0 \\ 0 & -13/17 & 31/17 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 18/17 \\ 20/17 \\ 154/17 \end{bmatrix}$$

Revised simplex table

	$\hat{B}^{-1}$				$\hat{x}_B$	$\hat{x}_5$	Mini ratio	
x_3	1	1	-2	0	2	$\boxed{2}$	1	$\rightarrow$
x_2	0	4/17	-3/17	0	18/17	3/17	6	
x_1	0	-5/17	8/17	0	20/17	-8/17	—	
	0	-13/17	31/17	1	154/17	-31/17		

$$\hat{B}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1 & 0 \\ -3/34 & 5/34 & 0 & 0 \\ 4/17 & -1/17 & 0 & 0 \\ 31/34 & 5/34 & 0 & 1 \end{bmatrix}$$

Net evaluation $z_j - c_j = \begin{bmatrix} \frac{31}{34} & \frac{5}{34} & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 & -1 & 1 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & \frac{31}{34} & \frac{5}{13} & 0 & M \end{bmatrix}$$

$$x_B = \hat{B}^{-1} b = \begin{bmatrix} 1/2 & 1/2 & -1 & 0 \\ -3/34 & 5/34 & 0 & 0 \\ 4/17 & -1/17 & 0 & 0 \\ 31/34 & 5/34 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} [illegible] \\ 30/34 \\ 28/17 \\ 185/17 \end{bmatrix}$$

$x_1 = \frac{28}{17}$, $x_2 = \frac{15}{17}$ Max. $Z = \frac{185}{17}$

Q4 Solve the transportation Problem

	W_1	W_2	W_3	W_4	
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4		I	II	III	IV	V
F_1	(5) 19	30	50	(2) 10	~~7~~ ~~2~~ 0	9	9	40	40	
F_2	70	30	(7) 40	(2) 60	~~9~~ 7	10	20	20	20	20
F_3	40	(8) 8	70	(10) 20	~~18~~ ~~10~~ 0	12	20	50		
	~~5~~ 0	~~8~~ 0	7	~~14~~ 4 ~~2~~ 0						
I	21	22	10	10						
II	21		10	10						
III			10	10						
IV			10	50						
V			40	60						

Now we Test for optimality

	W_1	W_2	W_3	W_4	
F_1	(5) 19	30 $\boxed{32}$	50 $\boxed{60}$	(2) 10	$v_1 = 10$
F_2	70 $\boxed{1}$	30 $\boxed{-18}$	(7) 40	(2) 60	$v_2 = 60$
F_3	40 $\boxed{11}$	(8) 8	70 $\boxed{70}$	(10) 20	$v_3 = 20$
	$u_1 = 9$	$u_2 = -12$	$u_3 = -20$	$u_4 = 0$	

No. of allocated cells are

$= m+n-1$

$= 3+4-1 = 6$

Net evaluation for cell (2, 2) is −18 ∴ solution is not optimum

Now we construct loop

(5) 19	30 [32]	50 [60]	10 (2)
70 [1]	30 [−18]	40 (7)	60 (2)
40 [11]	8 (8)	70 [70]	20 (10)

$2 + 0 = 2$
$2 - 2 = 0$
$2 + 10 = 12$
$2 - 8 = 6$

Improved solution is

	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$
$u_1 = 0$	19 (5)	30 [32]	50 [42]	10 (2)
$u_2 = 32$	70 [19]	30 (2)	40 (7)	60 [18]
$u_3 = 10$	40 [11]	8 (6)	70 [52]	20 (12)

all $d_{ij} \geq 0$ ∴ it is optimum level

cost of transporation is

$5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20 = 743$

Q5 Write the dual of the LPP

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

s.t.

$$4x_1 + 3x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 5x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

Solution :- first convert the given primal into standard form

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

$$4x_1 + 3x_2 + x_3 \leq 6$$

$$4x_1 + 3x_2 + x_3 \geq 6$$

$$x_1 + 2x_2 + 5x_3 \leq 4$$

$$x_1 + 2x_2 + 5x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

all the constraints must have $\leq$ sign, so

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

$$4x_1 + 3x_2 + x_3 \leq 6$$

$$-4x_1 - 3x_2 - x_3 \leq -6$$

$$x_1 + 2x_2 + 5x_3 \leq 4$$

$$-x_1 - 2x_2 - 5x_3 \leq -4$$

$$x_1, x_2, x_3 \geq 0$$

dual will be of form $\text{Min } Z_D = b^T W$

$$A^T W \geq C^T$$

$$W \geq 0$$

$$\text{Min } Z_D = 6w_1 - 6w_2 + 4w_3 - 4w_4$$

$$4w_1 - 4w_2 + w_3 - w_4 \geq 2$$

$$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$$

$$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$$

$$w_1, w_2, w_3, w_4 \geq 0$$

Let $w_1 - w_2 = w_1'$

$$w_3 - w_4 = w_2'$$

Now $\text{Min } Z_D = 6w_1' + 4w_2'$

$$4w_1' + w_2' \geq 2$$

$$3w_1' + 2w_2' \geq 3$$

$$w_1' + 5w_2' \geq 1$$

w_1' and w_2' are unrestricted in sign

Q6 Solve the given LPP graphically

$$\text{Max. } Z = 5x_1 + 3x_2$$

s.t.

$$3x_1 + 5x_2 \le 15$$

$$5x_1 + 2x_2 \le 10$$

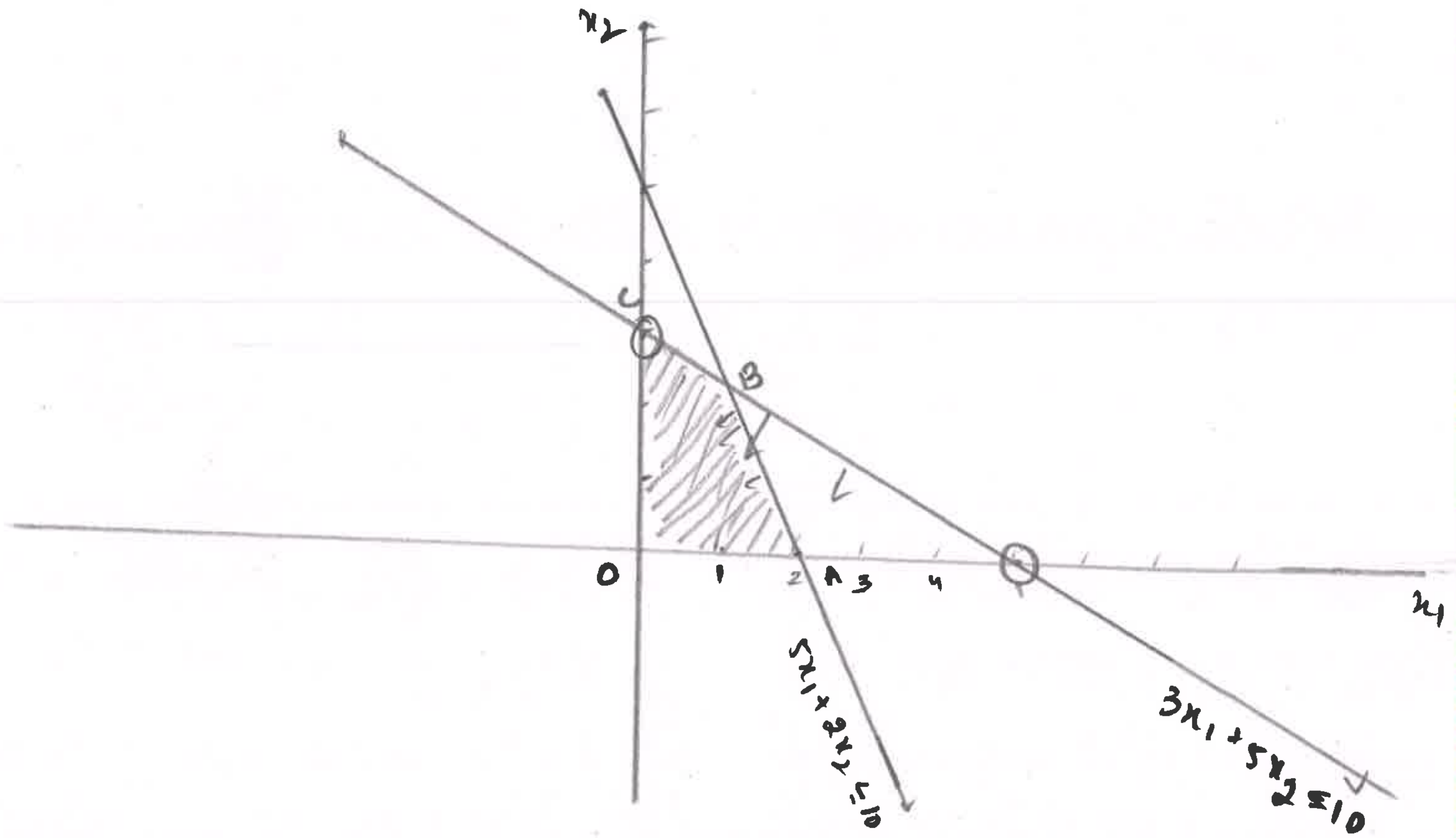
$$x_1, x_2 \ge 0$$

Solution

Draw lines corresponding to each constraints and find common region

$$\frac{x_1}{5} + \frac{x_2}{3} = 1$$

$$\frac{x_1}{2} + \frac{x_2}{5} = 1$$



Now calculate the objective function at corners

O (0 0)	$Z = 0$
A (2, 0)	$Z = 10$
B $\left(\frac{20}{19}, \frac{45}{19}\right)$	$Z = \frac{235}{19} \simeq 12.37$
C (0, 3)	$Z = 9$

optimum solution is

$x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

$$Z = \frac{235}{19}$$