Model Answers for B.Tech II Sem , Internal Assesment 1, Engg. Mathematics II, Branch Cs Section A&B

1(i) **§ 1.3. Rank.** ***Definition :*** *A non-negative integer r is said to be the rank of a matrix A, if it possesses the following properties :*

(*i*) *there is* ***atleast one minor*** *of order r, which does not vanish.*

(*ii*) *Every minor of order* $(r+1)$, *if any, vanishes.*

In fact (ii) $\Rightarrow$ every minor of order greater than r will vanish. It is so because every minor of order $(r+2)$, being the sum of multiples of minors of order $(r+1)$, will vanish.

Notation : The rank of a matrix A shall be denoted by the symbol $\rho\ (A)$.

Remark : The rank of a null matrix **O** of any order is defined as *zero.*

(ii) **§ 1.8. Harmonic Analysis:**

Some times in practice, the function is not given by a formula but by a graph or by a set of corresponding values. In such cases the Fourier coefficients a_0, a_n and b_n etc. can not be evalutated

Thus the process of finding the Fourier series when the function $f(x)$ is not given by an analytic axpression but only its numercial values are known at equally spaced points is known as Harmonic Analysis.

(iii)

§ 1.4. Dirichlet's Conditions :

A function $f(x)$ is expanded in the form of Fourier Series with the fo

assumptions :

(*i*) $f(x)$ is defined in $(-\pi, \pi)$ and is a single valued function .

(*ii*) $f(x)$ is a periodic function whose period is 2π .

(*iii*) The terms of the series can be integrated term by term *i.e.* the series is uniformly co

in the interval $(-\pi, \pi)$ and the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(*a*) converge to $f(x)$ when x is a point of continuity,

(*b*) converge to $\frac{f(x+0)+f(x-0)}{2}$, when x is a point of ordinary discontinuit

(*c*) converge to $\frac{f(-\pi+0)+f(\pi-0)}{2}$, when $x = \pm\pi$.

(iv)

$$f(x) = a_o + \sum_{n=1}^{\infty} \{a_n \cos\frac{2n\pi}{b-a}x + b_n \sin\frac{2n\pi}{b-a}x\}$$

$$\text{where } a_o = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi}{b-a}x\,dx$$

$$\&\ b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}x\,dx$$

2.

Ex. 1. *Obtain the constant term and the co-efficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table:*

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

[RTU B.Tech. (ME) 07; Raj. BE (CE), 05; (BT) 04; (EE) 03]

Sol. Let the Fourier series representing y in (0, 5) be

$$y = a_0 + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + \dots + b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3} + \dots, \quad \dots(1)$$

where $a_0 =$ [mean value of y in (0, 5); $\quad a_1 = 2\left[\text{mean value of } y\cos\frac{\pi x}{3} \text{ in } (0,5)\right]$

$$b_1 = 2\left[\text{mean value of } y\sin\frac{\pi x}{3} \text{ in } (0,5)\right]$$

The desired values are tabulated as follows :

x	$\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{\pi x}{3}$	y	$y\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$
0	0	0	1	9	0	9
1	$\pi/3$	$\sqrt{3}/2$	$\frac{1}{2}$	18	$9\sqrt{3}$	9
2	$2\pi/3$	$\sqrt{3}/2$	$-\frac{1}{2}$	24	$2\sqrt{3}$	−12
3	π	0	−1	28	0	−28
4	$4\pi/3$	$-\sqrt{3}/2$	$-\frac{1}{2}$	26	$-13\sqrt{3}$	−13
5	$5\pi/3$	$-\sqrt{3}/2$	$\frac{1}{2}$	20	$-10\sqrt{3}$	10
				$\Sigma = 125$	$-2\sqrt{3}$	−25

$$a_0 = \frac{125}{6} = 20.83; \qquad a_1 = -\frac{25}{3} = -8.33; \qquad b_1 = -\frac{2\sqrt{3}}{3} = -1.15$$

Substituting these values in (1) we get the required Fourier series as follows :

$$y = f(x) = 20.83 - 8.33\cos\frac{\pi x}{3} - 1.15\sin\frac{\pi x}{3} + \dots$$

3.

Ex. 1. *Find the Fourier series of the function* $f(x) = x + x^2$ *in the interval* $(-\pi, \pi)$ *and show that*

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Also find the sum of the series when $x = \pm\pi$.

[RTU B. Tech. III (EE), 07; Raj. BE (EIC, BME), 06; (BT) 04; (ECE),

Sol. Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Here $\quad f(x) = x + x^2, \quad -\pi < x < \pi$

Now $\quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x + x^2)\, dx$

$$a_0 = \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx \quad [\text{by } P_5]$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\left(\frac{\cos n\pi}{n^2}\right)\right] \quad [\because \sin n\pi = 0]$$

or $$a_n = \frac{4}{n^2}(-1)^n \quad \left[\because \cos n\pi = (-1)^n\right] \quad \dots(3)$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \quad [\text{by } P_5]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n}\right)+0\right]$$

or $$b_n = -\frac{2}{n}(-1)^n \quad \dots(4)$$

Now substituting the values of a_0, a_n and b_n from (2),(3) and (4) in (1) respectively, we obtain the required Fourier series as under ,

$$x+x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[(-1)^n\frac{4}{n^2}\cos nx + (-1)^n\left(-\frac{2}{n}\right)\sin nx\right]$$

or, $$x+x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \dots\right)$$
$$+2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right) \dots(5)$$

Replacing $x=\pi$ and $x=-\pi$ in (5),

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) \quad \dots(6)$$

and $$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) \quad \dots(7)$$

Adding (6) and (7),

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) \quad \text{or,} \quad \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

Sum of the series at $x = \pm\pi$

$$= \frac{1}{2}\left[f(-\pi+0) + f(\pi-0)\right] = \frac{1}{2}\lim_{h\to 0}\left[f(-\pi+h) + f(\pi-h)\right]$$

$$= \frac{1}{2}\lim_{h\to 0}\left[(-\pi+h) + (-\pi+h)^2 + (\pi-h) + (\pi-h)^2\right]$$

$$= \frac{1}{2}\left[-\pi+\pi^2+\pi+\pi^2\right] = \pi^2$$

4.

Example 42. Determine the values of a and b for which the system

$x + 2y + 3z = 6,\ x + 3y + 5z = 9,\ 2x + 5y + az = b$

has (i) no solution (ii) unique solution (iii) infinite number of solutions. Find the so in case (ii) and (iii).

Solution.

$$C = [A : B]$$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$, we have

$$C \simeq \left[\begin{array}{cccc} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & a-6 & : & 12 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$, we have

$$C \simeq \left[\begin{array}{ccccc} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & a-8 & : & b-15 \end{array}\right] \qquad .....(1)$$

Case (i) No solution : It is possible when $\rho(A) \neq \rho(C)$.

When $a - 8 = 0 \Rightarrow a = 8$ then $\rho(A) = 2$

When $b - 15 \neq 0 \Rightarrow b \neq 15$ then $\rho(C) = 3$

Thus for $a = 8$ and $b \neq 15$, $\rho(A) \neq \rho(C)$ and hence the system has no solution i.e., i is inconsistent.

Case (ii) Unique Solution: It is possible when $\rho(A) = \rho(C) = 3$ (number of unknowns).

For this, $a - 8 \neq 0 \Rightarrow a \neq 8$ and b can have any value so that $\rho(A) = 3 = \rho(C)$.

Further from (1), we have

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 3 \\ (a-8)z &= b - 15 \end{aligned}$$

or, $z = \dfrac{b-15}{a-8},\ y = \dfrac{3a-2b+6}{a-8},\ x = \dfrac{b-15}{a-8}$. **Ans**

Case (iii) Infinite Solutions: It is the case when $\rho(A) = \rho(C) < 3$.

Now when $a = 8$ and $b = 15$ then $\rho(A) = \rho(C) = 2$ (<3)

So for $a = 8$ and $b = 15$ the system has infinite solutions.

5. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Hence find A^{-1}. **[MREC 2002, MNIT-2005, 2006]**

Solution: *Statement:* Every square matrix A satisfies its own characteristic equation.
The characteristic equation for the given matrix is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda[(2-\lambda)(1-\lambda) - 3] - 1[1 - \lambda - 9] + 2[1 - 3(2 - \lambda)] = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 3\lambda - 1) + (\lambda + 8) + 2(3\lambda - 5) = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 8\lambda - 2 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 8\lambda + 2 = 0 \qquad .....(1)$$

which is the characteristic equation of A.

$$\text{Now } A^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix}$$

Now $A^3 - 3A^2 - 8A + 2I_3$

$$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - \begin{bmatrix} 21 & 12 & 15 \\ 33 & 24 & 33 \\ 12 & 18 & 30 \end{bmatrix} - \begin{bmatrix} 0 & 8 & 16 \\ 8 & 16 & 24 \\ 24 & 8 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Hence A satisfies its characteristic equation.

$\therefore$ The theorem is verified.

Since, $A^3 - 3A^2 - 8A + 2I_3 = 0$

$$\Rightarrow A^2 - 3A - 8I_3 + 2A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{-1}{2}[A^2 - 3A - 8I_3]$$

$$= \frac{-1}{2}\left(\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)$$

$$= \frac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$