14.4 Order of a differential equation

The **order of the highest order derivative** occurring in a differential equation is **called its order**

14.5 Degree of a differential equation :

The power of the highest order derivative in a differential equation is called its degree. Thus the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^4 + 3\left(\frac{dy}{dx}\right)^5 + y^2 = f(x) \qquad \text{is of order 2 and degree 4}$$

Radius of curvature ρ at any point on a curve is $\rho = \dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$ when freed from roo

it is expressed as $\left(\frac{d^2y}{dx^2}\right)^2 \rho^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3$

It is therefore a second order ordinary differential equation of degree two.

Ex. 3. Solve $\sin y \dfrac{dy}{dx} = \cos y\ (1 - x \cos y)$

Sol.: If we let $\cos y = z$

$$\sin y \frac{dy}{dx} = -\frac{dz}{dx},$$

the equation becomes $-\dfrac{dz}{dx} = z - x z^2$

or $-\dfrac{1}{z^2}\dfrac{dz}{dx} - \dfrac{1}{z} = -x$

if we let $\dfrac{1}{z} = u$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{du}{dx}$$

it becomes $\dfrac{du}{dx} - u = -x$

and it is now linear in u being of the form $\dfrac{du}{dx} + Py = Q$

So its I.F. $e^{\int P\,dx} = e^{\int -1\,dx} = e^{-x}$

Hence its solution is $u\ e^{-x} = -\int x\ e^{-x}\,dx + c$

$$= -\left(-x\,e^{-x} - e^{-x}\right) + c$$

or $\sec y = (x + 1) + c\,e^{x}$

Ex. 5. Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos x$

Sol.: The A.E. of (1) is $m^3 + 2m^2 + 1 = 0$

so $\quad (m^2 + 1)^2 = 0$ so $m = i, i, -i, -i$

Hence the C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

$$\text{P.I.} = \frac{1}{(D^2+1)^2} \text{ Real part of } x^2 e^{ix}$$

$$\text{P.I.} = \text{Re } e^{ix} \frac{1}{\left\{(D+i)^2 + 1\right\}^2} x^2$$

$$= \text{Re } e^{ix} \frac{1}{\left\{2iD + \left(1 + \frac{D}{2i}\right)\right\}^2} x^2$$

$$= \text{Re } \frac{-e^{ix}}{4D^2}\left(1 - \frac{iD}{2}\right)^{-2} x^2$$

$$= \text{Re } \frac{-e^{ix}}{4D^2}\left(1 - iD + \frac{3i^2D^2}{4}\right) x^2$$

$$= \text{Re } -\frac{1}{4} e^{ix} \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= \text{Re } -\frac{1}{4}(\cos x + i\sin x)\left(\frac{x^4}{12} + \frac{2i\,x^3}{6} - \frac{3}{2}\frac{x^2}{2}\right)$$

$$= -\frac{1}{4}\cos x\left(\frac{x^4}{12} - \frac{3x^2}{4}\right) + \frac{\sin x}{4}\frac{x^3}{3} \textbf{ Ans.}$$

Ex.4. Solve $$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+1y=\frac{1}{(1-x)^2}$$ ...(1) [RTU 2008]

[RTU 2009] [III Sem. E.C. 2005]

Sol.: This equation is both exact and also of the homogeneous form. Here we solve it by substitution $x = e^z$ or $z = \log x$.

Then $$x\frac{d}{dx}=\frac{d}{dz}=D$$

and $$x^2\frac{d^2}{dx^2}=D(D-1)$$

So the equation (1) becomes

$$\{D(D-1)+3D+1\}y=\frac{1}{(1-e^z)^2}$$ ...(2

It is now a second order differential equation with constant coefficients. Taking $y = e^{mz}$, it. A.E. is $m^2+2m+1=0$ so $m=-1, -1$ are its repeated roots.

So the C.F. $=(c_1+c_2z)e^{-z}$ and the P.I. $=\frac{1}{(D+1)(D+1)}\frac{1}{(1-e^z)^2}$

So by the general method

$$=e^{-z}\int e^z\, e^{-z}\left\{\int e^z\frac{1}{(1-e^z)^2}dz\right\}$$

$$=e^{-z}\int 1.\left[\frac{1}{1-e^z}\right]dz$$

$$=e^{-z}\int\frac{e^{-z}\,dz}{e^{-z}-1}=-e^{-z}\log(e^{-z}-1)$$

$$=-\frac{1}{x}\log\left(\frac{1}{x}-1\right)=\frac{1}{x}\log\frac{x}{1-x}$$

So the complete solution is $y =$ C. F. + P. I.

$$=\left\{\frac{1}{x}(c_1+c_2\log x)+\frac{1}{x}\log\frac{x}{1-x}\right\}$$

Ex. 1 Find the Fourier Series of the function $f(x) = x + x^2$ in the interval $(-\pi, \pi)$. Hence show that–

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots$$

Also find the sum of series when $x = \pm \pi$.

[BE 1985, 1990, 91, 92, 96, 2001]

Sol. Let us suppose the Fourier Series for the function $f(x) = x + x^2$ in the interval $(-\pi, \pi)$ is –

$$f(x) = x + x^2 = a_o + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \quad \ldots (1)$$

where a_o, a_n and b_n can be obtained as–

$$\because \quad a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \ldots (2)$$

$$\Rightarrow \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= 0+\frac{1}{2\pi}.2\int_0^{\pi}x^2dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\Rightarrow \quad a_0 = \frac{\pi^2}{3} \qquad \text{.... (3)}$$

$$\because \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx \qquad \text{.... (4)}$$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= 0+\frac{2}{\pi}= \int_0^{\pi}x^2\cos nx\,dx \qquad [\because x\cos nx \text{ is an odd function}]$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0+\frac{2\pi}{n^2}\cos n\pi + 0 + 0 + 0 + 0\right]$$

$$\Rightarrow \quad a_n = \frac{4}{n^2}(-1)^n \quad [\because \cos m\pi = (-1)^m] \qquad \text{.... (5)}$$

$$\because \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx \qquad \text{.... (6)}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx \qquad [\because x^2\sin nx \text{ is an odd function}]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi+0-0-0\right]$$

$$\Rightarrow \quad b_n = \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

or $\quad b_n = \frac{2}{n}(-1)^{n+1}$

Putting these values in (1), we have–

$$x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[(-1)^n \frac{4}{n^2}\cos nx + (-1)^{n+1}\frac{2}{n}\sin nx\right]$$

$$\Rightarrow \quad x + x^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\ldots\right]$$

$$+2\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots\ldots\right]$$

at $x = \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots\ldots\right]$$

$$+2\left[\sin\pi - \frac{1}{2}\sin 2\pi + \frac{1}{3}\sin 3\pi \ldots\ldots\right]$$

$$\Rightarrow \quad \pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

or $\quad \pi + \frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$

at $x = -\pi$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots\ldots\right]$$

$$+\left[-\sin\pi + \frac{1}{2}\sin 2\pi - \frac{1}{3}\sin 3\pi + \ldots\ldots\right]$$

$$\Rightarrow \quad -\pi + \frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

Adding (10) and (11) we have–

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

or $\quad \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$